Universidade do Minho
Departamento de Matemática

Primitivas

1. Comecemos por calcular $\int x^2 \operatorname{sen} x \, dx$.

   Sejam

   $$\begin{array}{ll} f'(x) = \operatorname{sen} x & f(x) = -\cos x \\ g(x) = x^2 & g'(x) = 2x \end{array}$$

   Aplicando o método de primitivação por partes, obtemos

   $$\int x^2 \operatorname{sen} x \, dx = -x^2 \cos x - \int -2x \cos x \, dx = -x^2 \cos x + \int 2x \cos x \, dx.$$

   Vamos aplicar de novo o método de primitivação por partes para calcular $\int 2x \cos x \, dx$.
   Sejam

   $$\begin{array}{ll} f'(x) = \cos x & f(x) = \operatorname{sen} x \\ g(x) = 2x & g'(x) = 2 \end{array}$$

   Aplicando o método de primitivação por partes, obtemos

   $$\int x^2 \operatorname{sen} x \, dx = -x^2 \cos x + 2\, x \operatorname{sen} x - \int 2 \operatorname{sen} x \, dx = -x^2 \cos x + 2\, x \operatorname{sen} x + 2\, \cos x + C, \qquad C \in \mathbb{R}.$$

   Vamos agora determinar o valor da constante C de modo a encontrar a primitiva que passa no ponto $(\frac{\pi}{2}, \pi)$. Tem-se que:

   $$-\frac{\pi^2}{4} \cos \frac{\pi}{2} + \pi \operatorname{sen} \frac{\pi}{2} + 2 \cos \frac{\pi}{2} + C = \pi \;\Leftrightarrow\; C = 0.$$

   Consequentemente, a primitiva que passa no ponto $(\frac{\pi}{2}, \pi)$ é

   $$F(x) = -x^2 \cos x + 2\, x \operatorname{sen} x + 2 \cos x.$$

2. (a) $f(x) = \dfrac{2x^3}{3} - \dfrac{x^2}{2} - 8x + \dfrac{65}{6}$.

   (b) $f(x) = \dfrac{5x}{4} - \dfrac{1}{8} \operatorname{sen}(2x)$.

3. [Primitivas imediatas]

(1) $$\int (\sqrt{x}+2)^2\,dx = \int (x+4\sqrt{x}+4)\,dx = \int x\,dx + \int 4\,x^{1/2}\,dx + \int 4\,dx$$
$$= \frac{x^2}{2} + 4\,\frac{x^{1/2+1}}{1/2+1} + 4x + C$$
$$= \frac{x^2}{2} + \frac{8}{3}\,x^{3/2} + 4x + C, \quad C \in \mathbb{R}$$

(2) $$\int (3x^2 - 2x^5)\,dx = x^3 - \frac{x^6}{3} + C, \quad C \in \mathbb{R}$$

(3) $$\int (2x+10)^{20}\,dx = \frac{1}{2}\int 2\,(2x+10)^{20}\,dx = \frac{1}{2}\frac{(2x+10)^{21}}{21} + C = \frac{(2x+10)^{21}}{42} + C, \quad C \in \mathbb{R}$$

(4) $$\int x^2 e^{x^3}\,dx = \frac{1}{3}\int 3\,x^2 e^{x^3}\,dx = \frac{1}{3}e^{x^3} + C, \quad C \in \mathbb{R}$$

(5) $$\int x^4 (x^5+10)^9\,dx = \frac{1}{5}\int 5\,x^4(x^5+10)^9\,dx = \frac{1}{5}\frac{(x^5+10)^{10}}{10} + C = \frac{(x^5+10)^{10}}{50} + C, \quad C \in \mathbb{R}$$

(6) $$\int \frac{2x+1}{x^2+x+3}\,dx = \ln(x^2+x+3) + C, \quad C \in \mathbb{R}$$

(7) $$\int \sqrt{2x+1}\,dx = \int (2x+1)^{1/2}\,dx = \frac{1}{2}\int 2\,(2x+1)^{1/2}\,dx = \frac{1}{2}\frac{(2x+1)^{3/2}}{3/2} + C$$
$$= \frac{1}{3}(2x+1)^{3/2} + C, \quad C \in \mathbb{R}$$

(8) $$\int \frac{x}{3-x^2}\,dx = -\frac{1}{2}\int \frac{-2x}{3-x^2}\,dx = -\frac{1}{2}\,\ln|3-x^2| + C, \quad C \in \mathbb{R}$$

(9) $$\int \frac{1}{4-3\,x}\,dx = -\frac{1}{3}\int \frac{-3}{4-3x}\,dx = -\frac{1}{3}\,\ln|4-3x| + C, \quad C \in \mathbb{R}$$

(10) $$\int \frac{1}{e^{3x}}\,dx = \int e^{-3x}\,dx = -\frac{1}{3}\int -3\,e^{-3x}\,dx = -\frac{1}{3}\,e^{-3x} + C, \quad C \in \mathbb{R}$$

(11) $$\int \frac{-7}{\sqrt{1-5x}}\,dx = \int -7\,(1-5x)^{-1/2}\,dx = \frac{-7}{-5}\int -5\,(1-5x)^{-1/2}\,dx$$
$$= \frac{7}{5}\,\frac{(1-5x)^{1/2}}{1/2} + C = \frac{14}{5}\,(1-5x)^{1/2} + C, \quad C \in \mathbb{R}$$

$$(12)\quad \int \frac{\sqrt{1+3\,\ln x}}{x}\,dx = \int \frac{1}{x}(1+3\,\ln x)^{1/2}\,dx = \frac{1}{3}\int \frac{3}{x}(1+3\,\ln x)^{1/2}\,dx$$
$$= \frac{1}{3}\,\frac{(1+3\,\ln x)^{3/2}}{3/2} + C = \frac{2}{9}\,(1+3\,\ln x)^{3/2} + C, \quad C\in\mathbb{R}$$

$$(13)\quad \int x\,\mathrm{sen}\,(x^2)\,dx = \frac{1}{2}\int 2x\,\mathrm{sen}\,(x^2)\,dx = -\frac{1}{2}\,\cos(x^2) + C, \quad C\in\mathbb{R}$$

$$(14)\quad \int \frac{1}{x(\ln^2 x+1)}\,dx = \int \frac{\frac{1}{x}}{1+\ln^2 x}\,dx = \mathrm{arctg}\,(\ln x) + C, \quad C\in\mathbb{R}$$

$$(15)\quad \int \left(\frac{2}{x}-3\right)^2 \frac{1}{x^2}\,dx = -\frac{1}{2}\int -\frac{2}{x^2}\left(\frac{2}{x}-3\right)^2\,dx$$
$$= -\frac{1}{2}\,\frac{\left(\frac{2}{x}-3\right)^3}{3} + C = -\frac{1}{6}\,\left(\frac{2}{x}-3\right)^3 + C, \quad C\in\mathbb{R}$$

$$(16)\quad \int \mathrm{sen}\,(\pi-2\,x)\,dx = -\frac{1}{2}\int -2\,\mathrm{sen}\,(\pi-2\,x)\,dx = \frac{1}{2}\,\cos(\pi-2\,x) + C, \quad C\in\mathbb{R}$$

$$(17)\quad \int \mathrm{th}\,x\,dx = \int \frac{\mathrm{sh}\,x}{\mathrm{ch}\,x}\,dx = \ln(\mathrm{ch}\,x) + C, \quad C\in\mathbb{R}$$

$$(18)\quad \int \mathrm{sen}\,x\,\cos x\,dx = \frac{\mathrm{sen}^2 x}{2} + C, \quad C\in\mathbb{R}$$

$$(19)\quad \int \mathrm{sen}\,(2x)\,\cos x\,dx = \int 2\,\mathrm{sen}\,x\cos^2 x\,dx = -2\int -\mathrm{sen}\,x\cos^2 x\,dx = \frac{-2\cos^3 x}{3} + C, \quad C\in\mathbb{R}$$

$$(20)\quad \int \mathrm{sen}^2\left(\frac{x}{2}\right)\,\cos^2\left(\frac{x}{2}\right)\,dx = \int \left(\frac{1-\cos x}{2}\right)\left(\frac{1+\cos x}{2}\right)\,dx = \frac{1}{4}\int (1-\cos x)(1+\cos x)\,dx$$
$$= \frac{1}{4}\int (1-\cos^2 x)\,dx = \frac{1}{4}\int \mathrm{sen}^2\,dx = \frac{1}{4}\int \frac{1-\cos(2x)}{2}\,dx$$
$$= \frac{1}{8}\int (1-\cos(2x))\,dx = \frac{1}{8}\int 1\,dx - \frac{1}{16}\int 2\,\cos(2x)\,dx$$
$$= \frac{1}{8}\,x - \frac{1}{16}\,\mathrm{sen}\,(2x) + C, \quad C\in\mathbb{R}$$

$$(21)\quad \int \mathrm{sen}^2 x\,dx = \int \frac{1-\cos(2x)}{2}\,dx = \frac{1}{2}\int 1\,dx - \frac{1}{4}\int 2\,\cos(2x)\,dx$$
$$= \frac{1}{2}\,x - \frac{1}{4}\,\mathrm{sen}\,(2x) + C, \quad C\in\mathbb{R}$$

(22) $$\int \cos^3\, dx = \int (1-\operatorname{sen}^2 x)\cos x\, dx = \int \cos x\, dx - \int \cos x \operatorname{sen}^2 x\, dx$$
$$= \operatorname{sen} x - \frac{\operatorname{sen}^3 x}{3} + C, \quad C \in \mathbb{R}$$

(23) $$\int \frac{x}{x^2-1}\, dx = \frac{1}{2}\int \frac{2x}{x^2-1}\, dx = \frac{1}{2}\ln|x^2-1| + C, \quad C \in \mathbb{R}$$

(24) $$\int \frac{x}{\sqrt{x^2-1}}\, dx = \frac{1}{2}\int 2x\,(x^2-1)^{-1/2}\, dx = \frac{1}{2}\,\frac{(x^2-1)^{1/2}}{1/2} + C = \sqrt{x^2-1} + C, \quad C \in \mathbb{R}$$

(25) $$\int \frac{1}{x}\operatorname{sen}(\ln x)\, dx = -\cos(\ln x) + C, \quad C \in \mathbb{R}$$

(26) $$\int \frac{-3}{x\,(\ln x)^3}\, dx = -3\int \frac{1}{x}\,(\ln x)^{-3}\, dx = -3\,\frac{(\ln x)^{-2}}{-2} + C = \frac{3}{2\,(\ln x)^2} + C, \quad C \in \mathbb{R}$$

(27) $$\int \frac{e^x}{1+e^{2x}}\, dx = \int \frac{e^x}{1+(e^x)^2}\, dx = \operatorname{arctg}(e^x) + C, \quad C \in \mathbb{R}$$

(28) $$\int \frac{e^x}{1-2e^x}\, dx = -\frac{1}{2}\int \frac{-2\,e^x}{1-2e^x}\, dx = -\frac{1}{2}\ln|1-2e^x| + C, \quad C \in \mathbb{R}$$

(29) $$\int \frac{1}{\cos^2(7x)}\, dx = \frac{1}{7}\int \frac{7}{\cos^2(7x)}\, dx = \frac{1}{7}\operatorname{tg}(7x) + C, \quad C \in \mathbb{R}$$

(30) $$\int \left(\sqrt{2x-1} - \sqrt{1+3x}\right) dx = \int (2x-1)^{1/2}\, dx - \int (1+3x)^{1/2}\, dx$$
$$= \frac{1}{2}\int 2\,(2x-1)^{1/2}\, dx - \frac{1}{3}\int 3\,(1+3x)^{1/2}\, dx$$
$$= \frac{1}{2}\,\frac{(2x-1)^{3/2}}{3/2} - \frac{1}{3}\,\frac{(1+3x)^{3/2}}{3/2} + C$$
$$= \frac{1}{3}(2x-1)^{3/2} - \frac{2}{9}(1+3x)^{3/2} + C, \quad C \in \mathbb{R}$$

(31) $$\int \frac{1}{x}\,(1+(\ln x)^2)\, dx = \int \frac{1}{x}\, dx + \int \frac{1}{x}(\ln x)^2\, dx$$
$$= \ln x + \frac{(\ln x)^3}{3} + C, \quad C \in \mathbb{R}$$

(32) $$\int \frac{2+\sqrt{\operatorname{arctg}(2x)}}{1+4x^2}\,dx = \int \frac{2}{1+(2x)^2}\,dx + \int \frac{1}{1+(2x)^2}(\operatorname{arctg}(2x))^{1/2}\,dx$$
$$= \int \frac{2}{1+(2x)^2}\,dx + \frac{1}{2}\int \frac{2}{1+(2x)^2}(\operatorname{arctg}(2x))^{1/2}\,dx$$
$$= \operatorname{arctg}(2x) + \frac{1}{2}\frac{(\operatorname{arctg}(2x))^{3/2}}{3/2} + C$$
$$= \operatorname{arctg}(2x) + \frac{1}{3}(\operatorname{arctg}(2x))^{3/2} + C, \quad C \in \mathbb{R}$$

(33) $$\int \frac{e^{\operatorname{arctg} x}}{1+x^2}\,dx = e^{\operatorname{arctg} x} + C, \quad C \in \mathbb{R}$$

(34) $$\int \frac{\operatorname{sen} x}{\sqrt{1+\cos x}}\,dx = \int \operatorname{sen} x\,(1+\cos x)^{-1/2}\,dx = -\int -\operatorname{sen} x\,(1+\cos x)^{-1/2}\,dx$$
$$= -\frac{(1+\cos x)^{1/2}}{1/2} + C = -2\sqrt{1+\cos x} + C, \quad C \in \mathbb{R}$$

4. [Primitivação por partes]

(1) Calcule $\int \ln x\,dx$.

Sejam

$$f'(x) = 1 \qquad f(x) = x$$
$$g(x) = \ln x \qquad g'(x) = \frac{1}{x}$$

Aplicando a fórmula de primitivação por partes, obtemos

$$\int \ln\,x\,dx = x\ln x - \int x\frac{1}{x}\,dx = x\ln x - \int 1\,dx = x\ln x - x + C, \quad C \in \mathbb{R}.$$

(2) Calcule $\int x\operatorname{sen}\,(2x)\,dx$.

Sejam

$$f'(x) = \operatorname{sen}(2x) \qquad f(x) = -\tfrac{1}{2}\cos(2x)$$
$$g(x) = x \qquad g'(x) = 1$$

Aplicando a fórmula de primitivação por partes, obtemos

$$\int x\operatorname{sen}\,(2x)\,dx = -\frac{1}{2}\,x\,\cos(2x) - \int -\frac{1}{2}\cos(2x).1\,dx = -\frac{1}{2}\,x\,\cos(2x) + \frac{1}{2}\int \cos(2x)\,dx$$
$$= -\frac{1}{2}\,x\,\cos(2x) + \frac{1}{4}\int 2\cos(2x)\,dx = -\frac{1}{2}\,x\,\cos(2x) + \frac{1}{4}\operatorname{sen}\,(2x) + C, \quad C \in \mathbb{R}.$$

(3) Calcule $\displaystyle\int \operatorname{arctg} x\, dx$.

Sejam

$$f'(x) = 1 \qquad f(x) = x$$
$$g(x) = \operatorname{arctg} x \qquad g'(x) = \frac{1}{1+x^2}$$

Aplicando a fórmula de primitivação por partes, obtemos

$$\begin{aligned}\int \ln\, x\, dx &= x \operatorname{arctg} x - \int \frac{x}{1+x^2}\, dx = x \operatorname{arctg} x - \frac{1}{2}\int \frac{2x}{1+x^2}\, dx \\ &= x \operatorname{arctg} x - \frac{1}{2}\ln(1+x^2) + C, \quad C \in \mathbb{R}.\end{aligned}$$

(4) Calcule $\displaystyle\int x \cos\, x\, dx$.

Sejam

$$f'(x) = \cos x \qquad f(x) = \operatorname{sen} x$$
$$g(x) = x \qquad g'(x) = 1$$

Aplicando a fórmula de primitivação por partes, obtemos

$$\int x \cos\, x\, dx = x \operatorname{sen} x - \int \operatorname{sen} x\, dx = x \operatorname{sen} x + \cos x + C, \quad C \in \mathbb{R}.$$

(5) Calcule $\displaystyle\int \ln(1-x)\, dx$.

Sejam

$$f'(x) = 1 \qquad f(x) = x$$
$$g(x) = \ln(1-x) \qquad g'(x) = \frac{-1}{1-x}$$

Aplicando a fórmula de primitivação por partes, obtemos

$$\begin{aligned}\int \ln(1-x)\, dx &= x\, \ln(1-x) - \int \frac{-x}{1-x}\, dx = x\, \ln(1-x) - \int \frac{1-x-1}{1-x}\, dx \\ &= x\, \ln(1-x) - \int 1\, dx + \int \frac{1}{1-x}\, dx \\ &= x\, \ln(1-x) - \int 1\, dx - \int \frac{-1}{1-x}\, dx \\ &= x\, \ln(1-x) - x - \ln(1-x)\, dx + C, \quad C \in \mathbb{R}.\end{aligned}$$

(6) Calcule $\displaystyle\int x\, \ln\, x\, dx$.

Sejam

$$f'(x) = x \qquad f(x) = \tfrac{x^2}{2}$$
$$g(x) = \ln x \qquad g'(x) = \tfrac{1}{x}$$

Aplicando a fórmula de primitivação por partes, obtemos

$$\begin{aligned}\int x \ln x\, dx &= \frac{x^2}{2}\ln x - \int \frac{x^2}{2}\frac{1}{x}\, dx = \frac{x^2}{2}\ln x - \int \frac{x}{2}\, dx \\ &= \frac{x^2}{2}\ln x - \frac{x^2}{4} + C, \quad C \in \mathbb{R}.\end{aligned}$$

(7) Calcule $\int x^2 \operatorname{sen} x\, dx$.

Sejam

$f'(x) = \operatorname{sen} x \qquad f(x) = -\cos x$

$g(x) = x^2 \qquad g'(x) = 2x$

Aplicando a fórmula de primitivação por partes, obtemos

$$\int x^2 \operatorname{sen} x\, dx = -x^2 \cos x - \int (-\cos x)2x\, dx = -x^2 \cos x + 2\int x \cos x\, dx.$$

Apliquemos de novo o método de primitivação por partes para calcular $\int x \cos x\, dx$.

Sejam

$f'(x) = \cos x \qquad f(x) = \operatorname{sen} x$

$g(x) = x \qquad g'(x) = 1$

Aplicando de novo a fórmula de primitivação por partes, obtemos

$$\begin{aligned}\int x^2 \operatorname{sen} x\, dx &= -x^2 \cos x + 2\int x \cos x\, dx \\ &= -x^2 \cos x + 2\left(x \operatorname{sen} x - \int \operatorname{sen} x\, dx\right) \\ &= -x^2 \cos x + 2\, x \operatorname{sen} x - 2\int \operatorname{sen} x\, dx \\ &= -x^2 \cos x + 2\, x \operatorname{sen} x + 2\cos x + C, \quad C \in \mathbb{R}.\end{aligned}$$

(8) Calcule $\int x \operatorname{sen} x \cos x\, dx$.

Sejam

$f'(x) = \operatorname{sen} x \cos x \qquad f(x) = \dfrac{\operatorname{sen}^2 x}{2}$

$g(x) = x \qquad g'(x) = 1$

Aplicando a fórmula de primitivação por partes, obtemos

$$
\begin{aligned}
\int x \operatorname{sen} x \cos x \, dx &= x\,\frac{\operatorname{sen}^2 x}{2} - \int \frac{\operatorname{sen}^2 x}{2}\, dx \\
&= x\,\frac{\operatorname{sen}^2 x}{2} - \frac{1}{2}\int \frac{1-\cos(2x)}{2}\, dx \\
&= x\,\frac{\operatorname{sen}^2 x}{2} - \int \frac{1}{4}\, dx + \frac{1}{4}\int \cos(2x)\, dx \\
&= x\,\frac{\operatorname{sen}^2 x}{2} - \int \frac{1}{4}\, dx + \frac{1}{8}\int 2\,\cos(2x)\, dx \\
&= x\,\frac{\operatorname{sen}^2 x}{2} - \frac{x}{4} + \frac{1}{8}\operatorname{sen}(2x) + C, \quad C \in \mathbb{R}.
\end{aligned}
$$

(9) Calcule $\int \ln^2 x \, dx$.
Sejam

$f'(x) = 1 \qquad f(x) = x$

$g(x) = \ln^2 x \qquad g'(x) = 2\,\frac{\ln x}{x}$

Aplicando a fórmula de primitivação por partes, obtemos

$$\int \ln^2 x \, dx = x\,\ln^2 x - \int 2x\,\frac{\ln x}{x}\, dx = x\,\ln^2 x - 2\int \ln x \, dx.$$

Pelo exercício 4.(1) temos que: $\int \ln x \, dx = x \ln x - x + C, \quad C \in \mathbb{R}$. Então,

$$\int \ln^2 x \, dx = x\,\ln^2 x - 2x \ln x + 2x + C, \quad C \in \mathbb{R}.$$

(10) Calcule $\int e^x \cos x \, dx$. Este exercício é análogo ao Exemplo 5 do ficheiro Aula 14 Cálculo (T). A solução deste exercício é:

$$\int e^x \cos x \, dx = \frac{e^x \cos x + e^x \operatorname{sen} x}{2} + C, \quad C \in \mathbb{R}.$$

(11) Calcule $\int \operatorname{arcsen} x \, dx$. Resolvido na aula TP.

(12) Calcule $\int e^{\operatorname{sen} x} \operatorname{sen} x \cos x \, dx$.
Sejam

$f'(x) = e^{\operatorname{sen} x} \cos x \qquad f(x) = e^{\operatorname{sen} x}$

$g(x) = \operatorname{sen} x \qquad g'(x) = \cos x$

Aplicando a fórmula de primitivação por partes, obtemos

$$
\begin{aligned}
\int e^{\operatorname{sen} x} \operatorname{sen} x \cos x \, dx &= e^{\operatorname{sen} x} \operatorname{sen} x - \int e^{\operatorname{sen} x} \cos x \, dx \\
&= e^{\operatorname{sen} x} \operatorname{sen} x - e^{\operatorname{sen} x} + C, \quad C \in \mathbb{R}.
\end{aligned}
$$

(13) Calcule $\displaystyle\int \frac{\operatorname{arcsen}\sqrt{x}}{\sqrt{x}}\,dx$.

Sejam

$$f'(x) = \tfrac{1}{\sqrt{x}} = x^{-1/2} \qquad f(x) = 2\sqrt{x}$$

$$g(x) = \operatorname{arcsen}\sqrt{x} \qquad g'(x) = \frac{\frac{1}{2\sqrt{x}}}{\sqrt{1-x}}$$

Aplicando a fórmula de primitivação por partes, obtemos

$$\begin{aligned}\int \frac{\operatorname{arcsen}\sqrt{x}}{\sqrt{x}}\,dx &= 2\sqrt{x}\operatorname{arcsen}\sqrt{x} - \int \frac{1}{\sqrt{1-x}}\,dx \\ &= 2\sqrt{x}\operatorname{arcsen}\sqrt{x} - \int (1-x)^{-1/2}\,dx \\ &= 2\sqrt{x}\operatorname{arcsen}\sqrt{x} + 2\sqrt{1-x} + C, \quad C \in \mathbb{R}.\end{aligned}$$

(14) Calcule $\displaystyle\int x \operatorname{arctg} x\,dx$. Resolvido na aula TP.

(15) Calcule $\displaystyle\int x^2 \log x\,dx$.

Sejam

$$f'(x) = x^2 \qquad f(x) = \tfrac{x^3}{3}$$

$$g(x) = \log x \qquad g'(x) = \tfrac{1}{x}$$

Aplicando a fórmula de primitivação por partes, obtemos

$$\begin{aligned}\int x^2 \log x\,dx &= \frac{x^3}{3}\log x - \int \frac{x^2}{3}\,dx \\ &= \frac{x^3}{3}\log x - \frac{x^3}{9} + C, \quad C \in \mathbb{R}.\end{aligned}$$

(16) Calcule $\displaystyle\int \operatorname{sen}(\log x)\,dx$.

Sejam

$$f'(x) = 1 \qquad f(x) = x$$

$$g(x) = \operatorname{sen}(\log x) \qquad g'(x) = \tfrac{1}{x}\cos(\log x)$$

Aplicando a fórmula de primitivação por partes, obtemos

$$\int \operatorname{sen}(\log x)\,dx = x\operatorname{sen}(\log x) - \int \cos(\log x)\,dx.$$

Apliquemos de novo o método de primitivação por partes para calcular $\displaystyle\int \cos(\log x)\,dx$.

Sejam

$f'(x) = 1 \qquad\qquad f(x) = x$

$g(x) = \cos(\log\, x) \qquad g'(x) = -\frac{1}{x}\text{sen}\,(\log\, x)$

Aplicando de novo a fórmula de primitivação por partes, obtemos

$$\begin{aligned}\int \text{sen}\,(\log\, x)\, dx &= x\,\text{sen}\,(\log\, x) - [x\, \cos(\log\, x) - \int -\text{sen}\,(\log\, x)\, dx] \\ &= x\,\text{sen}\,(\log\, x) - x\, \cos(\log\, x) - \int \text{sen}\,(\log\, x)\, dx.\end{aligned}$$

Então,

$$\int \text{sen}\,(\log\, x)\, dx = x\,\text{sen}\,(\log\, x) - x\, \cos(\log\, x) - \int \text{sen}\,(\log\, x)\, dx\,,$$

ou, de forma equivalente,

$$2\int \text{sen}\,(\log\, x)\, dx = x\,\text{sen}\,(\log\, x) - x\, \cos(\log\, x)\,.$$

Consequentemente,

$$\int \text{sen}\,(\log\, x)\, dx = \frac{x\,\text{sen}\,(\log\, x) - x\cos(\log\, x)}{2} + C, \quad C \in \mathbb{R}.$$

(17) Calcule $\int \text{ch}\, x\, \text{sen}\,(3x)\, dx$.

Sejam

$f'(x) = \text{ch}\, x \qquad\qquad f(x) = \text{sh}\, x$

$g(x) = \text{sen}\,(3x) \qquad g'(x) = 3\cos(3x)$

Aplicando a fórmula de primitivação por partes, obtemos

$$\int \text{ch}\, x\, \text{sen}\,(3x)\, dx = \text{sh}\, x\, \text{sen}\,(3x) - 3\int \text{sh}\, x \cos(3x)\, dx.$$

Apliquemos de novo o método de primitivação por partes para calcular $\int \text{sh}\, x\cos(3x)\, dx$.

Sejam

$f'(x) = \text{sh}\, x \qquad\qquad f(x) = \text{ch}\, x$

$g(x) = \cos(3x) \qquad g'(x) = -3\,\text{sen}\,(3x)$

Aplicando de novo a fórmula de primitivação por partes, obtemos

$$\begin{aligned}\int \text{ch}\, x\, \text{sen}\,(3x)\, dx &= \text{sh}\, x\, \text{sen}\,(3x) - 3[\text{ch}\, x\, \cos(3x) - \int -3\,\text{ch}\, x\, \text{sen}\,(3x)\, dx] \\ &= \text{sh}\, x\, \text{sen}\,(3x) - 3\,\text{ch}\, x\, \cos(3x) - 9\int \text{ch}\, x\, \text{sen}\,(3x)\, dx.\end{aligned}$$

Então,

$$\int \operatorname{ch} x \operatorname{sen}(3x)\, dx = \operatorname{sh} x \operatorname{sen}(3x) - 3 \operatorname{ch} x \, \cos(3x) - 9 \int \operatorname{ch} x \operatorname{sen}(3x)\, dx \,,$$

ou, de forma equivalente,

$$10 \int \operatorname{ch} x \operatorname{sen}(3x)\, dx = \operatorname{sh} x \operatorname{sen}(3x) - 3 \operatorname{ch} x \, \cos(3x)\,.$$

Consequentemente, $\displaystyle\int \operatorname{ch} x \operatorname{sen}(3x)\, dx = \frac{\operatorname{sh} x \operatorname{sen}(3x) - 3 \operatorname{ch} x \, \cos(3x)}{10} + C, \quad C \in \mathbb{R}.$

(18) Calcule $\displaystyle\int x^3 \, e^{x^2} \, dx$.

Sejam

$f'(x) = x\, e^{x^2}$ $\qquad f(x) = \frac{1}{2}\, e^{x^2}$

$g(x) = x^2$ $\qquad g'(x) = 2x$

Aplicando a fórmula de primitivação por partes, obtemos

$$\begin{aligned} \int x^3 \, e^{x^2} \, dx &= \frac{1}{2}\, x^2 \, e^{x^2} - \int x \, e^{x^2} \, dx \\ &= \frac{1}{2}\, x^2 \, e^{x^2} - \frac{1}{2}\, e^{x^2} + C, \quad C \in \mathbb{R}. \end{aligned}$$

5. [Primitivação por substituição]

(1) Exercício resolvido no Exemplo 1 do ficheiro Aula 15 Cálculo (T)

(2) Exercício análogo ao Exemplo 2 do ficheiro Aula 15 Cálculo (T)

(3) Calcule $\displaystyle\int \sqrt{4+x^2}\, dx$, efetuando a substituição $x = 2 \operatorname{sh} t, \quad t \geq 0$.

(i) <u>Susbtituição</u>:

Fazendo $x = 2 \operatorname{sh} t$, $t \geq 0$, tem-se que

$$\varphi(t) = 2 \operatorname{sh} t, \;\; \varphi'(t) = 2 \operatorname{ch} t, \;\; t = \operatorname{argsh}(x/2)\,.$$

(ii) <u>Cálculo da nova primitiva</u>:

$$\begin{aligned} \int \sqrt{4+x^2}\, dx &= \int \sqrt{4 + 4 \operatorname{sh}^2 t}.\, \underbrace{2 \operatorname{ch} t}_{\varphi'(t)} \, dt \\ &= \int 2\sqrt{1+\operatorname{sh}^2 t}.\, 2 \operatorname{ch} t \, dt = \int 4\sqrt{\operatorname{ch}^2 t}.\operatorname{ch} t \, dt = \int 4 \operatorname{ch}^2 t \, dt \\ &= 4 \int \frac{1+\operatorname{ch}(2t)}{2}\, dt = 4 \left(\frac{t}{2} + \frac{1}{4} \operatorname{sh}(2t) \right) + C, \quad C \in \mathbb{R} \\ &= 2t + \operatorname{sh}(2t) + C = 2t + 2 \operatorname{sh} t \operatorname{ch} t + C, \quad C \in \mathbb{R}. \end{aligned}$$

(iii) Regresso à variável inicial x:

Atendendo a que: $\operatorname{sh} t = \frac{x}{2}$, $t = \operatorname{argsh}(x/2)$ e a que

$$\operatorname{ch}^2 t - \operatorname{sh} t^2 = 1 \Rightarrow \operatorname{ch} t = \sqrt{1 + \operatorname{sh}^2 t} \quad (\operatorname{ch} t \geq 1)$$

obtemos que

$$\int \sqrt{4+x^2}\, dx = 2 \operatorname{argsh}(x/2) + x\sqrt{1+\frac{x^2}{4}} + C = 2 \operatorname{argsh}(x/2) + \frac{x}{2}\sqrt{4+x^2} + C, \quad C \in \mathbb{R}.$$

(4) Exercício resolvido no Exemplo 4 do ficheiro Aula 15 Cálculo (T)

(5) Exercício resolvido no Exemplo 3 do ficheiro Exercícios Resolvidos-3dezembro

(6) Exercício resolvido no Exemplo 3 do ficheiro Aula 15 Cálculo (T)

6. [Primitivação de funções racionais]

(1) Exercício resolvido no Exemplo 1 do ficheiro Aula 16 Cálculo (T)

(2) Exercício resolvido no Exemplo 3 do ficheiro Aula 16 Cálculo (T)

(3) Exercício resolvido no Exemplo 2 do ficheiro Aula 16 Cálculo (T)

(4) Exercício resolvido no Exemplo 4 do ficheiro Aula 16 Cálculo (T)

(5) $\displaystyle\int \frac{x^2-x+2}{x(x^2-2)}\, dx = -2\ln|x| + \ln|x-1| + 2\ln|x+1| + C, \quad C \in \mathbb{R}.$

(6) $\displaystyle\int \frac{27}{x^4-3\,x^3}\, dx = \frac{9}{2x^2} + \frac{3}{x} - \ln|x| + \log|x-3| + C, \quad C \in \mathbb{R}.$

(7) $\displaystyle\int \frac{x+3}{(x-2)\,(x^2-2\,x+5)}\, dx = \ln|x-2| - \frac{1}{2}\log|x^2-2x+5| + C, \quad C \in \mathbb{R}.$

(8) $\displaystyle\int \frac{x+1}{x(x^2+1)^2}\, dx = \ln|x| + \frac{1}{2(x^2+1)} - \frac{\ln(x^2+1)}{2} + \frac{\operatorname{arctg} x}{2} + \frac{x}{2(x^2+1)} + C, \quad C \in \mathbb{R}.$

7. (1) Calcule $\displaystyle\int \frac{1}{(2+\sqrt{x})^7\sqrt{x}} dx.$

$$\int \frac{1}{(2+\sqrt{x})^7\sqrt{x}}\, dx = \int \frac{1}{\sqrt{x}}(2+\sqrt{x})^{-7}\, dx = 2\int \frac{1}{2\sqrt{x}}(2+\sqrt{x})^{-7}\, dx$$

$$= 2\frac{(2+\sqrt{x})^{-6}}{-6} + C = -\frac{1}{3(2+\sqrt{x})^6} + C, \quad C \in \mathbb{R}.$$

(2) Calcule $\displaystyle\int \operatorname{tg}^2 x\, dx$.

$$\begin{aligned}\int \operatorname{tg}^2 x\, dx &= \int \frac{\operatorname{sen}^2 x}{\cos^2 x}\, dx = \int \frac{1-\cos^2 x}{\cos^2 x}\, dx \\ &= \int \frac{1}{\cos^2 x}\, dx - \int 1\, dx = \operatorname{tg} x - x + C, \quad C \in \mathbb{R}.\end{aligned}$$

(3) Calcule $\displaystyle\int \frac{x + (\operatorname{arcsen}(3x))^2}{\sqrt{1-9\,x^2}}\, dx$.

$$\begin{aligned}\int \frac{x + (\operatorname{arcsen}(3x))^2}{\sqrt{1-9\,x^2}}\, dx &= \int x(1-9x^2)^{-1/2}\, dx + \int \frac{1}{\sqrt{1-9\,x^2}} \left(\operatorname{arcsen}(3x)\right)^2\, dx \\ &= -\frac{1}{18}\int -18x(1-9x^2)^{-1/2}\, dx + \frac{1}{3}\int \frac{3}{\sqrt{1-(3x)^2}} \left(\operatorname{arcsen}(3x)\right)^2\, dx \\ &= -\frac{1}{18}\frac{(1-9x^2)^{1/2}}{1/2} + \frac{(\operatorname{arcsen}(3x))^3}{9} + C \\ &= -\frac{1}{9}\sqrt{1-9x^2} + \frac{(\operatorname{arcsen}(3x))^3}{9} + C, \quad C \in \mathbb{R}.\end{aligned}$$

(4) Calcule $\displaystyle\int \frac{1}{1+e^x}\, dx$.

$$\int \frac{1}{1+e^x}\, dx = \int \frac{1+e^x-e^x}{1+e^x}\, dx = \int 1\, dx - \int \frac{e^x}{1+e^x}\, dx = x - \ln(1+e^x) + C, \quad C \in \mathbb{R}.$$

(5) Calcule $\displaystyle\int \frac{1}{\cos^2 x\, \operatorname{sen}^2 x}\, dx$.

$$\begin{aligned}\int \frac{1}{\cos^2 x\, \operatorname{sen}^2}\, dx &= \int \frac{\cos^2 x + \operatorname{sen}^2 x}{\cos^2 x\, \operatorname{sen}^2 x}\, dx \\ &= \int \frac{1}{\operatorname{sen}^2}\, dx + \int \frac{1}{\cos^2 x}\, dx = -\operatorname{cotg} x + \operatorname{tg} x + C, \quad C \in \mathbb{R}.\end{aligned}$$

(6) Calcule $\displaystyle\int \frac{1}{x^2\sqrt{4-x^2}}\, dx$, efetuando a substituição $x = 2\operatorname{sen} t$.

(i) Susbtituição:

Fazendo $x = 2\operatorname{sen} t$, tem-se que

$$\varphi(t) = 2\operatorname{sen} t, \;\; \varphi'(t) = 2\cos t, \;\; t = \operatorname{arcsen}(x/2), \;\; t \in \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[.$$

(ii) Cálculo da nova primitiva:

$$\int \frac{1}{x^2\sqrt{4-x^2}}\,dx = \int \frac{1}{4\,\text{sen}^2 t\sqrt{4-4\text{sen}^2 t}}.\underbrace{2\cos t}_{\varphi'(t)}\,dt$$

$$= \int \frac{1}{8\,\text{sen}^2 t\;\cos t}.2\cos t\,dt$$

$$= \frac{1}{4}\int \frac{1}{\text{sen}^2 t}\,dt = -\frac{1}{4}\text{cotg}\,t + C, \quad C \in \mathbb{R}$$

(iii) Regresso à variável inicial x:

Atendendo a que:

$$x = 2\,\text{sen}\,t \quad \text{e a que} \quad 1 + \text{cotg}^2 t = \frac{1}{\text{sen}^2 t}$$

obtém-se que

$$\int \frac{1}{x^2\sqrt{4-x^2}}\,dx = -\frac{1}{4}\frac{\sqrt{4-x^2}}{x} + C, \quad C \in \mathbb{R}.$$